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COMPARING MATRIX ADJUSTMENT METHODS:  
A FORMAL APPROACH

Louis de Mesnard

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# Comparing Matrix Adjustment Methods: A Formal Approach <sup>1</sup>

Louis de Mesnard

*Regional Economics Application Laboratory*, University of Illinois at Urbana-Champaign, USA and University of Burgundy, Dijon, France, 2 Bd Gabriel, B.P. 26611, F-21066 Dijon Cedex, FRANCE. E-mail: *louis.de-mesnard@u-bourgogne.fr*.

**ABSTRACT.** Matrix adjustment methods are very useful in regional economics to project or update matrices. The principle consists of finding the matrix that is the closest to an initial matrix subject to column and row sum totals of a target matrix. Many authors have tried to determine which matrix-adjustment method is the best from an empirical point-of-view using real data. In order to address the question from a theoretical point-of-view, the article examines the simpler problem of vector adjustment and then returns consideration of matrices. The information-lost minimization (biproportional methods and RAS) leads to a multiplicative form and generalizes the linear model. On the other hand, the distance minimization that leads to an additive form tends to distort the data by giving a result asymptotically independent of the initial matrix. As a result, it is possible to conclude unambiguously that biproportional methods and RAS are the best for matrix adjustment because they generalize the linear model and are asymptotically the most respectful of the initial matrix while they do not generate surprising negative terms. Moreover, measuring the gap between the projection and the target to determine which method is the best is not a good idea because the gap depends on the data; on the contrary, the gap can be interpreted in terms of a structural effect that is a general form of the shift-share method.

**KEYWORDS.** Matrix adjustment, RAS, Biproportion, Input-Output.

**Abbreviated title.** Comparing matrix adjustment methods.

**JEL Classification.** C63, C67, D57.

## 1 Introduction

Matrix adjustment is intensively used in economics, but also in other fields. In economics, the most popular application is the elaboration in the System of National Accounts to obtain a matrix of interindustrial, regional, interregional or international flows. These matrix adjustment methods serve to equilibrate the accounting matrices or to derive present accounting matrices when they are unknown and when only present partial information (particularly about column and row sums) and past tables are given, or to make a forecast of future accounting matrices. Matrix adjustment methods can also be used to project a matrix in order to make it comparable to another matrix to identify the structural changes between both matrices.

The principle of matrix adjustment methods<sup>2</sup> consists of finding the matrix  $\hat{\mathbf{Z}}$  that is the closest to an initial matrix  $\mathbf{Z} \geq \mathbf{0}$  with respect of the margins (the column and row sum totals) of a target

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<sup>1</sup> Many thanks to Geoffrey Hewings for his useful remarks.

matrix  $\mathbf{Z}^* \geq \mathbf{0}$ ;  $\mathbf{Z}$  and  $\mathbf{Z}^*$  could be two matrices for two different countries or two matrices for the same country but for two different years (this paper is limited to the case of equality constraints with respect to the margins totals), expressed in the same units. The result is denoted  $\hat{\mathbf{Z}} = K(\mathbf{Z}, \mathbf{Z}^*)$ , where  $K$  is the method of matrix adjustment chosen to perform the operation. The most popular method of matrix adjustment is RAS, but many other methods can be used. These methods can be classified into two families: the so-called "additive" and "multiplicative" methods. The precise detailed differences will be developed later. Now it is sufficient to understand that  $\hat{\mathbf{Z}}$  obtained by additive methods is of the general form  $\mathbf{P} + \mathbf{Z} + \mathbf{Q}$ : in this category, the procedure that minimizes the distance between  $\mathbf{Z}$  and  $\hat{\mathbf{Z}}$  would be included. The general form of  $\hat{\mathbf{Z}}$  found by multiplicative methods is  $\mathbf{P} \mathbf{Z} \mathbf{Q}$ : in this last category, the well-known method RAS, also called biproportion, would be perhaps the most familiar technique. The authors who want to compare the merits of both methods<sup>3</sup> or to determine if one method is accurate (often: RAS)<sup>4</sup> tend to use real data such as international, national or regional data, then they make projections with both methods and they compare the results to decide which method is the best by measuring the gap between the projection and the target. This approach is empirical, that is, it is always limited by the data; by looking at empirical results, one is never sure that one method is better than another while the contrary could be true with different data. As many things have been said about the comparative merits and drawbacks of these methods from an empirical point-of-view, the aim of this paper is to decide which family of methods is the better from a theoretical point-of-view. In this paper, the match will be reduced to a duel between the two champions of each side, the minimization of distance and RAS. Considering the theoretical approach with matrices is not so easy, we will begin the exploration with vectors, thereby returning to a more simple problem, vector adjustment. A vector can be considered as a very simple case of a matrix, where one dimension is equal to 1, i.e.,  $(1, n)$  or  $(n, 1)$ . The case of vectors will also serve as a counter-example to dismiss some ideas often developed in matrix adjustment problems. However, as a matrix is not only a juxtaposition of vectors, the matrix

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<sup>2</sup> This paper is limited to non negative matrices; this excludes from the discussion GRAS (Junius and Oosterhaven, 2003). I will explain later why this choice has been made.

<sup>3</sup> See for example Jackson and Murray (2004).

adjustment problem is bi-dimensional, while the vector adjustment problem is mono-dimensional, it will be necessary to verify each time if the properties found for vectors are valid for matrices.

## 2 Matrix Adjustment

Two main families of methods can be listed to serve as the projector  $K$ , the additive methods (the projected matrix will be denoted  $\hat{\mathbf{Z}}^A$ , the projector  $K^A$ ) and the multiplicative methods (the projected matrix will be denoted  $\hat{\mathbf{Z}}^M$ , the projector  $K^M$ ). All must respect the same set of constraints:

$$\left\{ \begin{array}{l} \sum_{j=1}^m \hat{z}_{ij} = \sum_{j=1}^m z_{ij}^* \equiv z_{i\bullet}^* \text{ for all } i \\ \sum_{i=1}^n \hat{z}_{ij} = \sum_{i=1}^n z_{ij}^* \equiv z_{\bullet j}^* \text{ for all } j \end{array} \right\}$$

### 2.1 Additive Methods

Among additive methods, there is the minimization of the very familiar *Least Squares*, that is, the square of the distance between  $\hat{\mathbf{Z}}^A$  and  $\mathbf{Z}$  (Almon, 1968), a particular case of the Hölder

norm denoted  $\|\hat{\mathbf{Z}}^A - \mathbf{Z}\|_h = \left( \sum_{i=1}^n \sum_{j=1}^m |\hat{z}_{ij}^A - z_{ij}|^h \right)^{1/h}$ , where  $h = 2$ :

$$\min_{\hat{z}_{ij}^A} SS; SS = \sum_{i=1}^n \sum_{j=1}^m (\hat{z}_{ij}^A - z_{ij})^2, \text{ s.t. } \left\{ \begin{array}{l} \sum_{j=1}^m \hat{z}_{ij}^A \equiv z_{i\bullet}^* \text{ for all } i \\ \sum_{i=1}^n \hat{z}_{ij}^A \equiv z_{\bullet j}^* \text{ for all } j \end{array} \right\}$$

The solution is denoted (how it is derived is recalled in annex 1):

$$(1) \hat{\mathbf{Z}}^A = \mathbf{P} + \mathbf{Z} + \mathbf{Q} - w$$

$$\text{where } w = \frac{z_{\bullet\bullet}^* - z_{\bullet\bullet}}{n m}.$$

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<sup>4</sup> See for example Miernyk (1977), Hewings and Janson (1980), Miller and Blair (1985), Israilevich (1986), Szyrmer (1987), Lahr (2001) and de Mesnard and Miller (2006).

$\mathbf{P}$  and  $\mathbf{Q}$  are matrices of which form is  $\mathbf{P} = \begin{bmatrix} p_1 & \dots & p_1 \\ \dots & & \dots \\ p_n & \dots & p_n \end{bmatrix}$  and  $\mathbf{Q} = \begin{bmatrix} q_1 & \dots & q_m \\ \dots & & \dots \\ q_1 & \dots & q_m \end{bmatrix}$ , with:

$$(2) p_i = \frac{z_{i\bullet}^* - z_{i\bullet}}{m} \text{ for all } i=1,\dots,n \text{ and } q_j = \frac{z_{\bullet j}^* - z_{\bullet j}}{n} \text{ for all } j=1,\dots,m$$

A particular case is  $z_{\bullet\bullet}^* = z_{\bullet\bullet}$  so  $w = 0$  and  $\hat{\mathbf{Z}}^A = \mathbf{P} + \mathbf{Z} + \mathbf{Q}$ .

Other additive methods are:

| the Absolute Differences, which is the Hölder norm where  $h=1$  (this case is not differentiable, and must be solved by the simplex after separating the positive cases from the negative cases),

| the Weighted Absolute Differences (Lahr, 2001), i.e.,  $\min_{\hat{z}_{ij}^A} \sum_{i=1}^n \sum_{j=1}^m z_{ij} |\hat{z}_{ij}^A - z_{ij}|$ ,

| the Normalized Absolute Differences, i.e.,  $\min_{\hat{z}_{ij}^A} \sum_{i=1}^n \sum_{j=1}^m \frac{|\hat{z}_{ij}^A - z_{ij}|}{z_{ij}}$ ,

| the Weighted Square Differences, i.e.,  $\min_{\hat{z}_{ij}^A} \sum_{i=1}^n \sum_{j=1}^m z_{ij} (\hat{z}_{ij}^A - z_{ij})^2$ ,

| the Normalized Square Differences (Deming and Stephan, 1940), i.e.,  $\min_{\hat{z}_{ij}^A} \sum_{i=1}^n \sum_{j=1}^m \frac{(\hat{z}_{ij}^A - z_{ij})^2}{z_{ij}}$ ,<sup>5</sup>

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Most of these methods are not linear and need to be linearized (see also Senior and Wilson (1974)) before finding a solution by linear programming software but one must remember that linearization is not the panacea because it is only valid around the linearization point, assuming that the function is not so far from the hyperplane of linearization at the optimum.

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<sup>5</sup> See in Hewings and Janson (1980, pp. 852-3) how the solution of the four last methods can be derived.

<sup>6</sup> One can cite also Friedlander's method (1961) or the ASAM method (Durieux and Payen, 1976), used in the PROPAGE model by the INSEE (*Institut National de la Statistique et des Etudes Economiques* in France) to estimate the formation of the gross fixed-capital of firms in the French National Accounting (Hoh Ta Khanh, 1982, pp. 204-9).

## 2.2 Multiplicative Methods

Among multiplicative methods, there is RAS and other biproportional methods.<sup>7</sup> The RAS method (Stone, 1961; Stone and Brown, 1962; Stone, Bates and Bacharach, 1963) is often considered as appropriate for adjusting input-output tables (Fréchet, 1960; Froment and Lenclud, 1976), in order to project, update (Paelinck and Waelbroeck, 1963; Lynch, 1986), or estimate them (Allen and Lecomber, 1975; Thionet, 1976), and so forth (see also Sentis and Thionet, 1961; and Lecomber, 1975).<sup>8</sup> However, RAS was developed under the patronage of operations-research and it has the reputation for lacking theoretical foundations, even if Bacharach (1970) developed some theoretical properties (and particularly, the existence, uniqueness and convergence of the solution).<sup>9</sup>

The methods of this group can be derived by solving Kullback and Liebler's minimization of information loss (Kullback and Liebler, 1951; Kullback, 1959; Theil, 1967, 1971; Uribe, de Leeuw and Theil, 1965; Snickars and Weibull, 1977):

$$(3) \min_{\hat{z}_{ij}^M} I; I = \sum_{i=1}^n \sum_{j=1}^m \hat{z}_{ij}^M \log \frac{\hat{z}_{ij}^M}{z_{ij}}, \text{ s.t. } \left\{ \begin{array}{l} \sum_{j=1}^m \hat{z}_{ij}^M \equiv z_{i\bullet}^* \text{ for all } i \\ \sum_{i=1}^n \hat{z}_{ij}^M \equiv z_{\bullet j}^* \text{ for all } j \end{array} \right\}$$

generating (the derivation of the solution poses a difficulty; see annex 1):<sup>10</sup>

$$(4) \hat{\mathbf{Z}}^M = \mathbf{P} \mathbf{Z} \mathbf{Q}$$

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<sup>7</sup> The analysis is limited to matrices, that is, to tables of which elements are equipped with two indexes  $i$  and  $j$ . See Cole (1992) for the case with three indexes.

<sup>8</sup> However, from Miernyk (1977), RAS has been criticized because adding extra information can make things worse. After Miernyk's work, some other authors have also found such a problem: Miller and Blair (1985), Israilevich (1986), Szyrmer (1987) and Lahr (2001). See de Mesnard and Miller (2006) for a synthetic discussion and some recalculations.

<sup>9</sup> Stone and Leontief also...

<sup>10</sup> Alternately, it could be possible to solve  $\min_{\hat{z}_{ij}^M} I; I = \sum_{i=1}^n \sum_{j=1}^m z_{ij}^M \log \frac{z_{ij}^M}{\hat{z}_{ij}^M}$ , s.t.

$$\left\{ \begin{array}{l} \sum_{j=1}^m \hat{z}_{ij}^M \equiv z_{i\bullet}^* \text{ for all } i \\ \sum_{i=1}^n \hat{z}_{ij}^M \equiv z_{\bullet j}^* \text{ for all } j \end{array} \right\} \text{ but this requires making first a first-order linear approximation by using a}$$

$\mathbf{P}$  and  $\mathbf{Q}$  are diagonal matrices:

$$(5) \quad p_i = \frac{z_{i\bullet}^*}{\sum_{j=1}^m q_j z_{ij}}, \text{ for all } i, \text{ and } q_j = \frac{z_{\bullet j}^*}{\sum_{i=1}^n p_i z_{ij}} \text{ for all } j$$

It must be also noted that the system (5) is solved iteratively, for example:

$$(6) \quad p_i^{(t+1)} = \frac{z_{i\bullet}^*}{\sum_{j=1}^m q_j^{(t+1)} z_{ij}}, \text{ for all } i, \text{ and } q_j^{(t+1)} = \frac{z_{\bullet j}^*}{\sum_{i=1}^n p_i^{(t)} z_{ij}} \text{ for all } j.$$

The properties of equations (4) and (5) have been explored by Snickars and Weibull

(1977). The series  $p_i^{(t+1)} = z_{i\bullet}^* \left( \sum_{j=1}^m \frac{z_{\bullet j}^* z_{ij}}{\sum_{i=1}^n p_i^{(t)} z_{ij}} \right)^{-1}$  for all  $i$  have a fixed-point (the same type of

series can equivalently be derived for the  $q_j$ ). Unfortunately the fixed-point cannot be found analytically but only numerically by an iterative process: it is unique, existing and converging because the function  $I$  is a convex and continuously derivable function, defined on a compact set. Nevertheless, there are some cases where the solution could not exist;<sup>11</sup> each row agent must be connected to a sufficient number of column agents to be able to sell its production and conversely for column agents (they must be able to buy what they need). For example, consider

the matrix  $\mathbf{Z} = \begin{bmatrix} 5 & 0 \\ 4 & 3 \\ 9 & 3 \end{bmatrix} \begin{matrix} 5 \\ 7 \end{matrix}$ . The associated Boolean matrix derived from matrix  $\mathbf{Z}$  is  $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ :

there is no arc from vertex 1 to 2 so  $\hat{\mathbf{Z}}^M$  will necessarily have a zero in place (1, 2). Consider

the target matrix  $\mathbf{Z}^* = \begin{bmatrix} & 10 \\ & 2 \\ 7 & 5 \end{bmatrix}$ ; it is impossible to find  $\hat{\mathbf{Z}}^M$  because column agent 2 can buy

only 2 from itself while he needs 5; row agent 1 has to buy 10 but all of these 10 must go to

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Taylor expansion to fall on a  $\chi^2$  expression (Kadas and Klafsky, 1976). See a discussion in Hewings and Janson (1980, pp. 853-4).

<sup>11</sup> See Macgill (1977).

himself while he can absorb only 7. Except these impossible cases, which must be interpreted as the definition set of the function, the biproportional problem always has a solution.

Bachem and Korte (1979a and b) have demonstrated that equations (4) and (5) are computationally the most efficient but (4) and (5) can also be generated from the entropy maximization principle (Jaynes, 1957a and b; Wilson, 1970; Kapur, 1989):

$$\max_{\hat{z}_{ij}^M} \sum_{i=1}^n \sum_{j=1}^m \hat{z}_{ij}^M \log \hat{z}_{ij}^M, \text{ s.t. } \left\{ \begin{array}{l} \sum_{j=1}^m \hat{z}_{ij}^M = z_{i\bullet}^* \text{ for all } i \\ \sum_{i=1}^n \hat{z}_{ij}^M = z_{\bullet j}^* \text{ for all } j \\ \sum_{i=1}^n \sum_{j=1}^m \hat{z}_{ij}^M c_{ij} = C \end{array} \right.$$

where  $c_{ij}$  is some transaction cost involved in moving from  $i$  to  $j$  and  $C$  is the total transaction cost.<sup>12</sup> One passes from this model to the information loss minimization and vice-versa by

posing  $z_{ij} = \exp(-\gamma c_{ij}) \Leftrightarrow c_{ij} = -\frac{1}{\gamma} \log z_{ij}$ , denoted  $c_{ij} = k \log z_{ij}$  for all  $i, j$ , where  $\gamma$  is the

multiplier associated to the constraint of cost and  $k = -1/\gamma$  could be interpreted as a dimension factor.

The model can also be derived from the theory of gravitation (Nijkamp, 1975), the interaction minimization principle (Watanabe, 1969; Guiasu, 1979), the probabilistic multinomial model (Choukroun, 1975) and even by the theory of utility (Niedercorn and Bechdolt, 1969; Niedercorn and Moorhead, 1974). It is demonstrated that these algorithms generate (5) after some transformations (Vermot-Desroches, 1986; de Mesnard, 1988). Other methods are more or less closely related to these, such as the theory of movements (Alonso, 1978) (Ledent, 1981), Gumbel's method (Thionet, 1976), the TAU and UAT methods (Snower, 1970).

### 3 Vector Adjustment

A vector is also a matrix with only one column (or one row): not only does this make the problem simpler to handle but it also this helps clarify some conclusions visible with matrices. However, it is not possible to pass from the matrix adjustment problem to the vector adjustment problem simply by posing  $m = 1$  (or  $n = 1$ ) because the resulting problem would be trivial:



considering two vectors of real numbers,  $\mathbf{x} \in \mathfrak{R}^{n+}$  and  $\mathbf{x}^* \in \mathfrak{R}^{n+}$ , the set of constraints would

become  $\left\{ \begin{array}{l} \hat{x}_i = x_i^* \\ \sum_{i=1}^n \hat{x}_i = \sum_{i=1}^n x_i^* \end{array} \right\}$  for all  $i$ , and the only solution obviously would be  $\hat{x}_i = x_i^*$  for all  $i$ .

Hence, the problem of vector adjustment must be of the following form. Considering two vectors of real numbers, expressed in the same units,  $\mathbf{x} \in \mathfrak{R}^{n+}$  and  $\mathbf{x}^* \in \mathfrak{R}^{n+}$ , what vector  $\hat{\mathbf{x}} \in \mathfrak{R}^{n+}$  is the closest to a given "initial" vector  $\mathbf{x}$  in the sense of some criterion to be defined

where  $\hat{x}_\bullet = \sum_{i=1}^n \hat{x}_i$  is equal to  $x_\bullet^* \equiv \sum_{i=1}^n x_i^*$  ?

A very simple example could be the following: consider the vector  $\mathbf{x}' = (50 \ 30 \ 20)$ , the prime denoting the operation of transposition; its sum is equal to  $x_\bullet = 100$ . What is the vector  $\hat{\mathbf{x}}$  whose sum is equal to  $x_\bullet^* = 200$ ? Everyone, even a young schoolboy (after formulating the question in more familiar terms), would answer to a so simple question: it is the vector  $\hat{\mathbf{x}}' = (100 \ 60 \ 40)$ ! Everyone knows the answer because this is the *linear model*: one assumes that everything changes proportionally. However, it is not so simple. The question is: from where does this result come?

To facilitate a vector adjustment, we will consider the above two families of methods, the additive one and the multiplicative one. It will be demonstrated that, even if they could seem equivalent, only one allows for the derivation of the linear model, the other being affected by some serious drawbacks. All the derivations are very simple but they will be exposed completely to be sure that the reader is convinced by the rather surprising results.

### 3.1 Additive Formulation

The most common idea could be to use the very familiar *Least Squares* minimization, that is, of the square of the Euclidean distance in order to find the projected vector denoted  $\hat{\mathbf{x}}^A$ .

$$(7) \min_{\hat{x}_i^A} SS; SS = \sum_{i=1}^n (\hat{x}_i^A - x_i)^2, \text{ s.t. } \sum_{i=1}^n \hat{x}_i^A = x_\bullet^*$$

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<sup>12</sup> In commodity flow matrices, this cost will normally be the transportation cost but it could also be a generalized cost involving non line-haul charges (e.g., waiting, transfer costs)

where  $x_{\bullet}^* = \sum_i x_i^*$ . Criterion (7) amounts to finding the orthogonal projection (or the quadratic

mean) created by Gauss. Minimizing the distance or its squares is the same thing (apart from the fact that one has to discard the case  $SS = 0$ ): as  $\min_{\hat{x}_i^A} \sqrt{SS}$  devolves to computing

$$\frac{d\sqrt{SS}}{d\hat{x}_i^A} = 0 \Leftrightarrow \frac{1}{2\sqrt{SS}} \frac{dSS}{d\hat{x}_i^A} = 0, \text{ the derivatives of } SS \text{ and } \sqrt{SS} \text{ have the same zeros (but one}$$

must impose  $SS \neq 0$  when  $\sqrt{SS}$  is minimized).<sup>13</sup> More formally, given two functions  $g$  and  $f$ ,  $\min (g \circ f)$  is identical to  $\min f$  as soon as  $g$  is a monotonous increasing function (the symbol  $\circ$  denotes the composition of functions); here  $f$  is the sum of the squares and  $g$  is the square root. To summarize, a distance and its square have the same minimum, even if the square of a distance is not a distance. The solution is (see annex 1):

$$(8) \hat{x}_i^A = b + x_i \text{ for all } i, \text{ or } \hat{\mathbf{x}}^A = \mathbf{b} + \mathbf{x}$$

where the fixed term  $b = \frac{x_{\bullet}^* - x_{\bullet}}{n}$  is added to  $x_i$  to obtain  $\hat{x}_i^A$  and  $\mathbf{b}' = b(1 \dots 1)$ . The projected point  $\hat{\mathbf{x}}^A$  is the orthogonal projection of  $\mathbf{x}$ .

Going back to the above example,  $b = \frac{200-100}{3} = 33.33\dots$ , so the projected vector is

$$\hat{\mathbf{x}}^A = 33.33\dots + \mathbf{x}' = (83.33\dots \ 63.33\dots \ 53.33\dots).$$

Two other examples (a consumption function and a production function) are provided in annex 2. In figure 1,  $A$  is the initial point,  $B$  is the projection of  $A$  on the subspace  $DE$  defined by the constraint; the distance between  $A$  and  $B$  is smaller than the distance between  $A$  and any point different to  $B$  on the straight line  $DE$ .

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Figure 1 about here

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### 3.2 *Multiplicative Formulation*

With the multiplicative formulation, the projected vector denoted  $\hat{\mathbf{x}}^M$  could be obtained from the minimization of the information loss:

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<sup>13</sup> In other words, a distance is not differentiable at the origin but the sum squares is always differentiable.

$$(9) \min_{\hat{x}_i^M} I; I = \sum_{i=1}^n \hat{x}_i^M \log \frac{\hat{x}_i^M}{x_i}, \text{ s.t. } \sum_{i=1}^n \hat{x}_i^M = x^*$$

The solution is:

$$(10) \hat{x}_i^M = a x_i \text{ for all } i, \text{ or } \hat{\mathbf{x}}^M = \mathbf{a} \mathbf{x}$$

where  $a = \frac{x^*}{x}$  is fixed and is multiplied by  $x_i$  to obtain  $\hat{x}_i^M$ .

Again, going back to the example,  $a = \frac{200}{100} = 2$ , so the projected vector is  $\hat{\mathbf{x}}^M = 2 \mathbf{x} = (100 \ 60 \ 40)$  as suggested above. See annex 2 for other examples. In figure 2, the origin  $O$ , the initial point  $A$  and the projection  $C$  are aligned.

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Figure 2 about here

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Note that the generalization of the vector adjustment problem as a matrix adjustment problem is not the above matrix adjustment problem. For example, the generalization of the multiplicative vector adjustment problem is the proportional matrix adjustment problem.

#### 4 Comparative Properties of the Methods *4.1 Asymptotic Properties*

It is obvious that any projector must retain its properties when both matrices or both vectors move away, that is, when one matrix or vector goes to infinity, i. e., when the projection subspace goes far from the initial point (or when the line  $DE$  goes far from the origin toward  $D'E'$  and beyond to infinity). In the world of vectors, one or more of the coordinates of vector  $\mathbf{x}^*$  tend toward infinity; considering the examples in annex 2, this could mean that the consumer's wealth or the output of the firm are infinitely increased. In the world of matrices, this could mean that the margins of  $\mathbf{Z}^*$  are much larger than those of  $\mathbf{Z}$ :  $\mathbf{Z}$  and  $\mathbf{Z}^*$  describe an economy for two very distant years or two very different countries. If a method is not asymptotically correct, it could be dangerous to apply it. By considering the asymptotic properties of the estimator, one avoids using a method that may have some undesirable properties.

### 4.1.1 Vectors

Property 1. When  $x_i^*$  tends toward infinity, with the orthogonal projector, projected point's slopes,  $\frac{\hat{x}_j^A}{\hat{x}_i^A}$  for all  $i, j$ , tend to be equal to 1, so  $\lim_{x_i^* \rightarrow \infty} \hat{\mathbf{x}}^{A'} = k \mathbf{s}$ , with  $k$  arbitrarily large, and the projected point tends to become independent of the initial point of the problem,  $\mathbf{x}$  (of which slopes are equal to  $\frac{x_j}{x_i}$  for all  $i, j$ ). Conversely, with the proportional projector, slopes do not

change when  $x_i^*$  tend toward infinity:  $\frac{\hat{x}_j^M}{\hat{x}_i^M} = \frac{x_j}{x_i}$  for all  $i, j$  always and  $\lim_{x_i^* \rightarrow \infty} \hat{\mathbf{x}}^{M'} = k \mathbf{x}$ , with  $k$

arbitrarily large: projections  $\hat{\mathbf{x}}^M$  are aligned.

In figure 1, the projected point  $B'$  tend to go on the line  $Of$  (of which slope is equal to 1) when  $D'E'$  tend to infinity; in figure 2, the projected points  $C$  and  $C'$  remain on the same straight line  $OA$  passing through the origin, even asymptotically.

Proof.

Equation (9) shows that when  $x_i^*$  tends toward infinity, that is to say, when the hyperplane

$\sum_{i=1}^n x_i^* = x_i^*$  at infinity, is far from the hyperplane  $\sum_{i=1}^n x_i = x_i$ . (both hyperplanes' slopes are equal to

-1), the quantity  $b$  tends toward infinity, as the quantities  $\hat{x}_j^A$  do. So, all slopes  $\frac{\hat{x}_j^A}{\hat{x}_i^A} = \frac{x_j + b}{x_i + b}$  tend

toward 1 for all  $i, j$  (as the terms  $x_i$  tend to be very small facing to  $b$ ) On the contrary, equation

(10) immediately shows that the two vectors  $\hat{\mathbf{x}}^M$  and  $\mathbf{x}$  are, and remain, collinear: the quantity  $a$

tends toward infinity as the quantities  $\hat{x}_j^M$  but  $\frac{\hat{x}_j^M}{\hat{x}_i^M} = \frac{x_j}{x_i}$  for all  $i, j$ .  $\mathbf{x}$

In the above example,  $\mathbf{x}' = (50 \ 30 \ 20)$  and  $x_i^* \rightarrow \infty$ ; so  $\lim_{x_i^* \rightarrow \infty} \hat{\mathbf{x}}^{A'} = k (1 \ 1 \ 1)$  but

$\lim_{x_i^* \rightarrow \infty} \hat{\mathbf{x}}^{M'} = k (.5 \ .3 \ .2)$ , with  $k$  arbitrarily large:  $\hat{\mathbf{x}}^A$  becomes independent of  $\mathbf{x}$  (even if it remains

the closest to  $\mathbf{x}$  in terms of distance) but  $\hat{\mathbf{x}}^M$  remains linearly linked to  $\mathbf{x}$ .

### 4.1.2 Matrices

Let us return to the matrices. To study the asymptotic properties, considers that matrix  $\mathbf{Z}^*$  that goes "far from" matrix  $\mathbf{Z}$ .

Theorem 1. Assume that all  $z_{i\bullet}^*$  and  $z_{\bullet j}^*$  tend toward infinity. The projected matrix  $\hat{\mathbf{Z}}^A$  tends to be asymptotically uniform and independent of the initial matrix  $\mathbf{Z}$ :  $\lim_{\substack{z_{i\bullet}^* \rightarrow \infty \\ z_{\bullet j}^* \rightarrow \infty}} \hat{\mathbf{Z}}^A = k \mathbf{S}$ , with  $k$

arbitrarily large, where  $\mathbf{S}$  is the unit matrix, that is,  $\mathbf{S} = \begin{bmatrix} 1 & \dots & 1 \\ \dots & & \dots \\ 1 & \dots & 1 \end{bmatrix}$ . The projected matrix  $\hat{\mathbf{Z}}^M$

tends to remain asymptotically biproportional to  $\mathbf{Z}$ :  $\lim_{\substack{z_{i\bullet}^* \rightarrow \infty \\ z_{\bullet j}^* \rightarrow \infty}} \hat{\mathbf{Z}}^M = k \mathbf{P} \mathbf{Z} \mathbf{Q}$ , with  $k$  arbitrarily large

and  $\mathbf{P}$  and  $\mathbf{Q}$  finite.

Theorem 1 means that the orthogonal projection is asymptotically meaningless but also it means that even when we are not at infinity, data are distorted with additive methods.

Proof.

For the additive method, assume that the margins  $z_{i\bullet}^*$  and  $z_{\bullet j}^*$  tend toward infinite at the speed 1

so  $z_{\bullet\bullet}^*$  tends toward infinite at the speed  $nm$  and  $w = \frac{z_{\bullet\bullet}^* - z_{\bullet\bullet}}{nm}$  tends toward infinity at the speed

1;  $p_i = \frac{z_{i\bullet}^* - z_{i\bullet}}{m}$  and  $q_j = \frac{z_{\bullet j}^* - z_{\bullet j}}{n}$  tend also toward infinity at the speed 1, for all  $i, j$ . Hence,

from (1), each term  $\hat{z}_{ij}^A$  also tends toward infinity at the speed 1; consequently any slopes  $\frac{\hat{z}_{ij}^A}{\hat{z}_{ij}^A}$  for

all  $i$  or  $\frac{\hat{z}_{i'j}^A}{\hat{z}_{i''j}^A}$  for all  $j$  tend toward 1. Finally  $\lim_{\substack{z_{i\bullet}^* \rightarrow \infty \\ z_{\bullet j}^* \rightarrow \infty}} \hat{\mathbf{Z}}^A = k \mathbf{S}$ , with  $k$  arbitrarily large.

For the multiplicative method, assume that all the margins  $z_{i\bullet}^*$  and  $z_{\bullet j}^*$ , for all  $i, j$ , are multiplied by  $k$  arbitrarily large, that is,  ${}^L z_{i\bullet}^* = k z_{i\bullet}^*$  for all  $i$  and  ${}^L z_{\bullet j}^* = k z_{\bullet j}^*$  for all  $j$ . Thus, following (6), i.e., if the initialization of the iterative process is done by the  $p, q_j^{(t+1)}$  is multiplied by  $k$ , that is,

${}^L q_j^{(t+1)} = k q_j^{(t+1)}$  but  ${}^L p_i^{(t+1)} = p_i^{(t+1)}$  for all  $i$ ; if the initialization is such that the terms  $q$ , the terms  $p$  are multiplied by  $k$  and the terms  $q$  remain fixed. Finally, remembering that biproportion is hyperbolically homogenous, i.e., if the  $p$  are all multiplied by *any*  $\delta$ , the  $q$  are all divided by  $\delta$ , one has  ${}^L \hat{z}_{ij}^M = k \hat{z}_{ij}^M \Leftrightarrow {}^L p_i^{(t+1)} z_{ij} {}^L q_j^{(t+1)} = k p_i^{(t+1)} z_{ij} q_j^{(t+1)}$ , which tends toward infinity if  $k \rightarrow \infty$  but remains asymptotically biproportional to  $\mathbf{Z}$  and the ratio  $\frac{{}^L \hat{z}_{ij}^M}{{}^L \hat{z}_{ij}^M}$  for all  $i$  and

$$\frac{{}^L \hat{z}_{i'j}^M}{{}^L \hat{z}_{i''j}^M} \text{ for all } j \text{ remain unchanged: } \frac{{}^L \hat{z}_{ij}^M}{{}^L \hat{z}_{ij}^M} = \frac{\hat{z}_{ij}^M}{\hat{z}_{ij}^M} \text{ for all } i \text{ and } \frac{{}^L \hat{z}_{i'j}^M}{{}^L \hat{z}_{i''j}^M} = \frac{\hat{z}_{i'j}^M}{\hat{z}_{i''j}^M} \text{ for all } j. \text{ x}$$

## 4.2 Negative Terms

### 4.2.1 Vectors

Property 2. In some cases, the orthogonal projector could generate some negative components on the projected vector  $\hat{\mathbf{x}}^A$ , while this is impossible with the proportional projector for  $\hat{\mathbf{x}}^M$ . In the world of vectors, this implies some difficulties for most economic applications, such as negative inputs for production functions or negative uses for consumption functions, etc. (see the examples in annex 2).

Proof.

For the orthogonal projector, how  $b$  works in (9) obviously does not depend on the fact that (7) is written, either  $\min_{\hat{x}_i^A} \sum_{i=1}^n (\hat{x}_i^A - x_i)^2$  or  $\min_{\hat{x}_i^A} \sum_{i=1}^n (x_i - \hat{x}_i^A)^2$ . However, according to (8), as  $b$  could be negative as soon as  $x_\bullet^* < x_\bullet$ , some terms in  $x_i^*$  could be negative while some other terms remain non negative. More precisely, a negative component will appear in (8) as soon as  $\min_i x_i < -b \Leftrightarrow x_\bullet^* < x_\bullet - n \left( \min_i x_i \right)$ .

For the proportional projector, the proof is immediate from (10):  $a > 0$  so  $\hat{x}_i^M > 0$  for all  $i$ . x

Returning to the above example, when  $x_\bullet^* = 50$ ,  $b = \frac{1}{3}(50 - 100) = -16.66\dots$  so  $\hat{\mathbf{x}}^A = (33.33\dots \ 13.33\dots \ 3.33\dots)$ : all components are still nonnegative. If

$x_{\bullet}^* = x_{\bullet} - n \left( \min_i x_i \right) = 100 - 3 \times 20 = 40$ , then  $b = \frac{1}{3} (40 - 100) = -20$  so  $\hat{\mathbf{x}}^A = (30 \ 10 \ 0)$ :  $\hat{x}_3^A = 0$ ;  
but when  $x_{\bullet}^* = 10$ ,  $b = \frac{1}{3} (10 - 100) = -30$  so  $\hat{\mathbf{x}}^A = (20 \ 0 \ -10)$ :  $\hat{x}_3^A$  is negative.

### 4.2.2 Matrices

Returning to matrices, it is known that additive methods may generate some negative terms in  $\hat{\mathbf{Z}}^A$  even if they are not present in  $\mathbf{Z}$  (Thionet, 1976). A negative  $\hat{z}_{ij}^A$  is surprising if the corresponding  $z_{ij}$  is not negative. In matrices, following graph theory, they are reversed arcs; so if  $z_{ij}$  is the flow of commodity  $i$  sold by sector  $i$  to sector  $j$ , what could be  $\hat{z}_{ij}^A$  if it is negative? It is possible to introduce a set of non-negativity constraints (Froment and Lenclud, 1976) but the solutions will tend to accumulate themselves on the borders of the convex constraint set (instead of being negative, a  $\hat{z}_{ij}^A$  will become to be equal to zero); again, this is unrealistic.<sup>14</sup>

Biproportional methods guarantee the non negativity of the solution; if there are no negatives in  $\mathbf{Z}$ , there will be no negatives in  $\hat{\mathbf{Z}}^M$  (a positive is projected as a positive, a zero as a zero). This is clearly an important advantage.<sup>15</sup>

The following obvious theorem holds:

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<sup>14</sup> If  $\mathbf{Z}$  and  $\mathbf{Z}^*$  are non negative one must not accept negative terms in the projection. Why? First, if it is always hard to give an economic interpretation to negative terms in  $\mathbf{Z}$  (e.g., what is the meaning of a negative flow of steel from Steel industry to Cars industry). It is difficult to explain how a positive can turn out to be a negative after projection (e.g., how a positive flow of steel from Steel industry to Cars industry would become a negative flow of steel). It cannot be the flow of commodity  $i$  sold by sector  $j$  to sector  $i$ . For example, if the Cars industry have shells of old cars to be resold to Steel industry, this flow will be lower than the flow of steel sold by the Steel industry to the Cars industry; the flow will remain non negative, while it is not the Cars industry that is in charge of recycling shells but it is, generally, the Service industry. And it cannot be the flow of commodity  $j$  sold by sector  $j$  to sector  $i$ , a completely nonsense transmutation!

This is why negative matrices have not been considered while negative terms obviously could appear in negative matrices.

<sup>15</sup> It must be noted that recently Junius and Oosterhaven (2003) have introduced a special version of RAS (Generalized-RAS) that allows handling  $\mathbf{Z}$  matrices with negative terms: a positive  $z_{ij}$  is projected as a positive, a zero is projected as a zero and a negative is projected as a negative; the important fact is that no positive can be projected as a negative. In that sense, GRAS guarantee the non-switching from positive to negative and vice-versa. See also Oosterhaven (2005).

Theorem 2. The orthogonal projector could generate some negative components on the projected matrix  $\hat{\mathbf{Z}}^A$ , while it is impossible with the biproportional projector for the projected matrix  $\hat{\mathbf{Z}}^M$ . Proof. The proof is well known.

In equations (1) and (2), it is sufficient to have  $z_{i\bullet}^* < z_{i\bullet}$  to obtain  $p_i < 0$  for any  $i$ , or it is sufficient to have  $z_{\bullet j}^* < z_{\bullet j}$  to obtain  $q_j < 0$  for any  $j$ . Hence,  $\mathbf{P}$  and  $\mathbf{Q}$  could have negative components and  $\hat{z}_{ij}^A$  could be negative for any  $i$  or  $j$ .

In equations (4) and (5), iteratively solved, if no  $p_i^{(t)}$  are negative, all  $q_j^{(t+1)}$  are not negative, so no  $p_i^{(t+1)}$  are negative. As the solution is always unique, existing, converging and hyperbolically homogenous, one can initialize by any set of  $p_i^{(0)}$ . It is sufficient to choose  $p_i^{(0)} > 0$  for all  $i$  to be sure that all  $p_i^{(t)}$  and  $q_j^{(t)}$  are not negative. Thus,  $\hat{\mathbf{Z}}^M = \mathbf{P} \mathbf{Z} \mathbf{Q}$  is not negative.

#### 4.3 Interpreting the Gap between the Projection and the Target

##### 4.3.1 Vectors

When one moves to vectors, two vectors,  $\mathbf{x}$  the initial vector and  $\mathbf{x}^*$  the target vector, must be compared. Considering figures 1 and 2, one has to compare the position of the projections  $B$  (representing  $\hat{\mathbf{x}}^A$ ) and  $C$  (representing  $\hat{\mathbf{x}}^M$ ) of the point  $A$  by respect to any point  $F$  placed on the straight line  $DE$ ;  $F$  represents the target vector  $\mathbf{x}^*$ . Assume that  $F$  is **arbitrary**, as soon as it is placed on the straight line  $DE$ . The quantities  $|\hat{\mathbf{x}}^A - \mathbf{x}^*|$  (distance  $FB$ ) and  $|\hat{\mathbf{x}}^M - \mathbf{x}^*|$  (distance  $FC$ ) are the gaps generated by the additive and multiplicative methods respectively (see figure 3).

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Figure 3 about here

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Property 3. The gap generated by the proportional (that is, linear) projector could be equal, lower or larger than the gap generated by the orthogonal projector: the quantity  $|\hat{\mathbf{x}}^M - \mathbf{x}^*| - |\hat{\mathbf{x}}^A - \mathbf{x}^*|$  could be positive, zero or negative depending on the value of  $\mathbf{x}^*$ .



Proof.

Consider the trivial case where  $\mathbf{x}^* = \mathbf{x}$  (that is,  $F$  is on the line  $GH$  supporting  $A$ ; it is a projection on itself). The points  $B$  and  $C$  are confused with  $A$  so  $FB = FC$  and  $|\hat{\mathbf{x}}^M - \mathbf{x}^*| = |\hat{\mathbf{x}}^A - \mathbf{x}^*|$ .

Now, consider the non trivial case where  $\mathbf{x}^* \neq \mathbf{x}$  (that is,  $F$  is not on the line  $GH$  that supports  $A$ ).

| Consider the sub-case where  $A$  is on the first bisector: in this case,  $A$ ,  $B$  and  $C$  are aligned on this bisector and  $FB = FC$ , so  $|\hat{\mathbf{x}}^M - \mathbf{x}^*| = |\hat{\mathbf{x}}^A - \mathbf{x}^*|$ .

Consider the sub-case where  $A$  is not on the first bisector, so  $BC > 0$ .

P If  $A$  is to the left of the first bisector,  $C$  is to the left of  $B$ ; there are three cases:

-  $F$  is between  $E$  and  $B$ . The distance  $FC$  is always larger than the distance  $FB$  because

$$\vec{FC} = \vec{FB} + \vec{BC} \text{ and } FC > FB.$$

-  $F$  is between  $B$  and  $C$ . The value of  $FB$  and  $FC$  depends on the exact position of  $F$ , all is possible:  $FC > FB$ ,  $FC = FB$  and  $FC < FB$ .

-  $F$  is between  $C$  and  $D$ .  $\vec{FB} = \vec{FC} + \vec{CB}$ , so  $FC < FB$ .

If  $A$  is to the right of the first bisector,  $C$  is to the right of  $B$ . There are again the same three cases, reversed.

-  $F$  is between  $D$  and  $B$ ; the distance  $FC$  is always larger than the distance  $FB$  because

$$\vec{FC} = \vec{FB} + \vec{BC} \text{ and } FC > FB.$$

-  $F$  is between  $B$  and  $C$ . The value of  $FB$  and  $FC$  depends on the exact position of  $F$ , all is possible.

-  $F$  is between  $C$  and  $E$ ;  $\vec{FB} = \vec{FC} + \vec{CB}$ , so  $FC < FB$ .

Two interpretations of the gap between the projection and the target can be produced:

| Either the gap reflects the quality of the projection and this gap, considered as a bias, must be minimized. From Property 3, the interpretation of the gap as a bias that must be as small as possible in order to maintain the quality of the projection ("accuracy" of the method) cannot be

accepted, because the comparison of both methods on the basis of true data depends entirely on the nature of these data,  $\mathbf{x}$  and  $\mathbf{x}^*$ , hence, the comparison can be favorable to one or to the other.

| Or the gap measures the structural effect between the initial situation and the final situation. When the projector is linear, this second interpretation in terms of structural change corresponds, more or less, in the world of vectors, to the familiar *shift-share method*:  $|\hat{\mathbf{x}} - \mathbf{x}^*|$  measures the *structural effect*, while the *size effect* is removed by calculating  $\hat{\mathbf{x}}$  (Armstrong and Taylor, pp. 145-7).

### 4.3.2 Matrices

The gap between the projected matrix  $\hat{\mathbf{Z}} = K(\mathbf{Z}, \mathbf{Z}^*)$  and the target matrix  $\mathbf{Z}^*$  is  $|K(\mathbf{Z}, \mathbf{Z}^*) - \mathbf{Z}^*|$ .

Note that how the gap is measured is not essential in the following discussion (, see Knudsen and Fotheringham (1986) for a discussion of the choice of the "statistics" affects the measure of the gap; they list three categories of "statistics")

| information-based "statistics" (as Kullback and Liebler's *information gain* function

$$\phi = \sum_{i=1}^n \sum_{j=1}^m \left| \hat{z}_{ij}^M \log \frac{\hat{z}_{ij}^M}{z_{ij}} \right|, \text{ or as } \psi = \sum_{i=1}^n \sum_{j=1}^m \left( \hat{z}_{ij}^M \log \frac{\hat{z}_{ij}^M}{z_{ij} + \hat{z}_{ij}^M} + z_{ij} \log \frac{z_{ij}}{z_{ij} + \hat{z}_{ij}^M} \right),$$

general distance "statistics" (as *standardized root mean error*)

and the traditional statistics (as  $R^2$  or  $\chi^2$ ).

One must not confuse between these "statistics" that calculate the distance between two matrices (in the context of this article, it is  $K(\mathbf{Z}, \mathbf{Z}^*)$  and  $\mathbf{Z}^*$ ), and the way that  $K(\mathbf{Z}, \mathbf{Z}^*)$  is generated; critically, this distinction is important for the criterion of information gain.

For matrices, the gap could receive two interpretations.

#### *Interpreting the Gap as a Bias*

The projection method is declared as inaccurate if the bias is large; the authors supporting this point of view tend to prefer a projection method that minimizes the bias.

Theorem 3. Both families of methods could provide the smallest bias: it is sufficient to choose the adequate target matrix  $\mathbf{Z}^*$ .

Proof. For  $K^A$  as well as for  $K^M$ , the projected matrix  $\hat{\mathbf{Z}} = K(\mathbf{Z}, \mathbf{Z}^*)$  depends only on the margins  $\mathbf{Z}^* \mathbf{s}$  and  $\mathbf{s}' \mathbf{Z}^*$  of  $\mathbf{Z}^*$ , not on the interior values  $z_{ij}^*$  of  $\mathbf{Z}^*$  itself (if the margins are determined by the interior values, the contrary is false). With the margins being given, any method could provide a small bias; it is sufficient to choose the interior of  $\mathbf{Z}^*$ , that is, the adequate set of  $z_{ij}^*$ , the margins  $\mathbf{Z}^* \mathbf{s}$  and  $\mathbf{s}' \mathbf{Z}^*$  being given.

Among these matrices  $\mathbf{Z}^*$ , there is always one that provides the smallest bias.

| When the projector is  $K^M$  (equations 3 and 4), it is  $\hat{\mathbf{Z}}^M = K^M(\mathbf{Z}, \mathbf{Z}^*)$  itself because  $K^M(\mathbf{Z}, \hat{\mathbf{Z}}^M) = \hat{\mathbf{Z}}^M$ ; so, if the set of data is  $(\mathbf{Z}, \hat{\mathbf{Z}}^M)$ , the bias is zero:  $\hat{\mathbf{Z}}^M - K^M(\mathbf{Z}, \hat{\mathbf{Z}}^M) = 0$ .

| When the projector is  $K^A$  (equations 1 and 2), it is  $\hat{\mathbf{Z}}^A = K^A(\mathbf{Z}, \mathbf{Z}^*)$  itself because  $K^A(\mathbf{Z}, \hat{\mathbf{Z}}^A) = \hat{\mathbf{Z}}^A$ , hence, if the set of data is  $(\mathbf{Z}, \hat{\mathbf{Z}}^A)$ , the bias is zero:  $\hat{\mathbf{Z}}^A - K^A(\mathbf{Z}, \hat{\mathbf{Z}}^A) = 0$ .

Consequently, if by chance  $\mathbf{Z}^*$  is equal to (or is close to)  $\hat{\mathbf{Z}}^M$ , one will find that biproportion is the best method, while if  $\mathbf{Z}^*$  is equal to (or is close to)  $\hat{\mathbf{Z}}^A$  one will find that the minimization of distance is the best method.

For example, measuring the performances of two methods by using the  $n \times m$  table of year 1980 as initial matrix and the  $n \times m$  table of year 2003 as target matrix is not theoretically correct because an infinity of tables could have the same margins as the table of 2003. One method could be the best with one table but the worse with another table, the margins being given, so the results are entirely dependent of what nature of the economy is in 2003. Epistemologically, these results are not universal but *contingent* on the empirical data. The choice of another year could reverse the results. One might consider all tables from 1981 until 2003; if all the results or the majority of the results are in favor of one of the methods, this one could be promoted as the best. However, even if this approach could seem to be empirically acceptable (except that the tables in this series of tables will not have the same margins), it is not theoretically correct. The series of tables 1981, 1982, ..., 2003, etc., is not the whole set of possible matrices and it would be necessary to consider the infinite set of *all* possible matrices of dimensions  $n \times m$  and the results could be different with one or more of the matrices (realistic or not) of this set.

It must be said that this finding is not generally appreciated. In most works, a set of data is used to validate or invalidate a method or the other, forgetting that the results are strictly contingent to the data, that is, absolutely not general.<sup>16</sup> As any change in the initial matrix (or in the margins) obviously implies a different result,<sup>17</sup> this result seems obvious for RAS. However, the demonstration in this present paper goes far from this: for **any method**, comparing the results of the projection  $\hat{\mathbf{Z}} = K(\mathbf{Z}, \mathbf{Z}^*)$  to  $\mathbf{Z}^*$  is contingent to the data inside  $\mathbf{Z}^*$ .

### *Interpreting the Gap as the Structural Change*

Interpreting the gap as the *structural change* between  $\mathbf{Z}$  and  $\mathbf{Z}^*$  was suggested initially by de Mesnard (1988, 1997, 2004):<sup>18</sup>  $\hat{\mathbf{Z}}^M = K^M(\mathbf{Z}, \mathbf{Z}^*)$  and  $\mathbf{Z}^*$  are comparable as they have the same margins  $z_{i\cdot}^*$  for all  $i$  and  $z_{\cdot j}^*$  for all  $j$ . Computing  $\hat{\mathbf{Z}}^M = K^M(\mathbf{Z}, \mathbf{Z}^*)$  allows to remove the effect of variation of the size (i.e., of differential growth) of the agents  $i$  and  $j$  (which could be sectors, regions, individuals, etc.) between  $\mathbf{Z}$  and  $\mathbf{Z}^*$ . As in the shift-share method, it remains the structural effect measured by the norm  $\|\hat{\mathbf{Z}}^M - \mathbf{Z}^*\|$ , which can be globally computed, or computed

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<sup>16</sup> Some authors, mainly Miernyk (1977), have adopted this point of view, not to compare additive and multiplicative methods but to criticize RAS: they introduce some new information (one or more elements, exogenously given) and they discover that the bias is sometimes increased. Some other authors as Hewings and Janson (1977), Miller and Blair (1985), Israilevich (1986), Szyrmer (1987) and Lahr (2001) find similar results. However, de Mesnard and Miller (2006) have recomputed Myernik's calculations and some of its followers to find rather different results: adding extra information improves the results more often than previously found. Anyway, theorem 3 again applies, as suggested in the conclusion of de Mesnard and Miller (2006): extra information is able to improve or not the results of RAS: it is sufficient to choose skillfully the target matrix.

<sup>17</sup> Bacharach (1970) has demonstrated that  $\hat{\mathbf{Z}}^M = K^M(\mathbf{Z}, \mathbf{Z}^*) = \mathbf{P} \mathbf{Z} \mathbf{Q}$ , ( $\mathbf{P}$  and  $\mathbf{Q}$  diagonal matrices), is unique when it is found by RAS, only given the initial matrix and the future margins  $\mathbf{Z}^* \mathbf{s}$  and  $\mathbf{s}' \mathbf{Z}^*$  of  $\mathbf{Z}^*$ . More generally, de Mesnard (1994) has demonstrated that the same problem has a unique solution whatever the algorithm chosen even unknown (and not only for RAS or for the minimization of information (3)-(5) or for the maximization of entropy or for any other); in other words, all algorithms are equivalent from the moment that they allow to find  $\hat{\mathbf{Z}}^M = K^M(\mathbf{Z}, \mathbf{Z}^*) = \mathbf{P} \mathbf{Z} \mathbf{Q}$ : this is the property of "unicity".

<sup>18</sup> See also Van der Linden and Dietzenbacher (1995, 2000), Dietzenbacher and Hoekstra (2003); Dietzenbacher and Hoekstra call the gap the "cell-specific change".

column vector by column vector, or computed row vector by row vector.<sup>19</sup> Considering  $\hat{\mathbf{Z}}^M - \mathbf{Z}^* = K^M(\mathbf{Z}, \mathbf{Z}^*) - \mathbf{Z}^*$ , one should note that if the value of  $\hat{\mathbf{Z}}^M = K^M(\mathbf{Z}, \mathbf{Z}^*)$  does not depend on the interior values  $z_{ij}^*$  of  $\mathbf{Z}^*$  (but only on the interior values  $z_{ij}$  of  $\mathbf{Z}$  and on the margins of  $\mathbf{Z}^*$ ), the structural change between  $\mathbf{Z}$  and  $\mathbf{Z}^*$  depends on these interior values of  $\mathbf{Z}^*$ ! The same thing could be suggested with  $K^A$ , but negative terms could appear in the structural change: again, how to interpret a negative structural change?

## 5 Conclusion

To help deciding which matrix-adjustment method is the best, the article returned to the simpler problem of vector adjustment before addressing matrices. It has been shown that the information-lost minimization (biproportional methods and RAS) leads to a multiplicative form and generalizes the linear model. On the other hand, the distance minimization that leads to an additive form tends to distort the data by giving a result that is asymptotically independent (all its components being equal) to the initial matrix. The result suggest unambiguously that biproportional methods and RAS are the best for matrix adjustment as they generalize the linear model and are asymptotically the most respectful of the initial matrix while the do not generate surprising negative terms. The demonstration in this paper is theoretical while most authors have tried to provide an empirical demonstration by measuring the gap between the projected matrix and the target matrix but the empirical demonstration is strictly contingent on the data and absolutely not general. Hence measuring the gap between the projection and the target cannot help deciding which method is the best because the gap depends on the chosen target matrix. On the contrary, the same gap could be interpreted in terms of structural effect generalizing the shift-share method.

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<sup>19</sup> Theil and Ghosh (1980) have also made a parallel between shift-share and RAS. However, it is by using a completely different definition of shift-share and of its generalization to matrices, what amounts to say that the shift-share equation (in matrices) is  $\hat{\mathbf{Z}}^M = K^M(\mathbf{Z}, \mathbf{Z}^*)$  itself. Another story.

## 1 Annexes

### 1.1 Annex 1: Derivation of the solutions of both methods of matrix adjustment

#### 1.1.1 Solution of distance minimization

For vectors, the Lagrangian writes as:

$$L = \sum_{i=1}^n (\hat{x}_i^A - x_i)^2 + \lambda \left( x_{\bullet}^* - \sum_{i=1}^n \hat{x}_i^A \right)$$

The first derivatives are all zero at optimum:

$$\frac{dL}{d\hat{x}_i^A} = 0 \text{ for all } i \Leftrightarrow 2(\hat{x}_i^A - x_i) - \lambda = 0 \text{ for all } i \Leftrightarrow$$

$$(11) \quad \hat{x}_i^A = x_i + \frac{\lambda}{2} \text{ for all } i$$

$$\frac{dL}{d\lambda} = 0 \Leftrightarrow x_{\bullet}^* = \sum_{i=1}^n \hat{x}_i^A$$

So, by replacing  $\hat{x}_i^A$  from (11) into this equation:  $x_{\bullet}^* = x_{\bullet} + \frac{n\lambda}{2} \Leftrightarrow \lambda = \frac{2}{n}(x_{\bullet}^* - x_{\bullet})$ ; then by

carrying  $\lambda$  into (11):  $\hat{x}_i^A = b + x_i$  for all  $i$ .

For matrices, the Lagrangian writes as:

$$L = \sum_{i=1}^n \sum_{j=1}^m (\hat{z}_{ij}^A - z_{ij})^2 + \lambda_i \left( z_{i\bullet}^* - \sum_{j=1}^m \hat{z}_{ij}^A \right) + \mu_j \left( z_{\bullet j}^* - \sum_{i=1}^n \hat{z}_{ij}^A \right)$$

The derivatives equalized to zero at optimum give:

$$\frac{dL}{d\hat{z}_{ij}^A} = 0 \Leftrightarrow 2(\hat{z}_{ij}^A - z_{ij}) - \lambda_i - \mu_j = 0$$

$$(12) \quad \hat{z}_{ij}^A = z_{ij} + \frac{1}{2}(\lambda_i + \mu_j) = 0 \text{ for all } i, j$$

Then,

$$\frac{dL}{d\lambda_i} = 0 \Leftrightarrow z_{i\bullet}^* = \sum_{j=1}^m \hat{z}_{ij}^A \text{ for all } i$$

By inserting the value of  $\hat{z}_{ij}^A$  from (12) in this equation, it follows:

$$z_{i\bullet}^* = z_{i\bullet} + \frac{1}{2} \sum_{j=1}^m (\lambda_i + \mu_j) \Leftrightarrow \lambda_i = \frac{2(z_{i\bullet}^* - z_{i\bullet}) - \mu_{\bullet}}{m} \text{ and } m \lambda_{\bullet} = 2(z_{\bullet\bullet}^* - z_{\bullet\bullet}) - n \mu_{\bullet}.$$

And,

$$\frac{dL}{d\mu_j} = 0 \Leftrightarrow z_{\bullet j}^* = \sum_{i=1}^n \hat{z}_{ij}^A \text{ for all } j$$

and again by inserting (13):

$$z_{\bullet j}^* = z_{\bullet j} + \frac{1}{2} \sum_{i=1}^n (\lambda_i + \mu_j) \Leftrightarrow \mu_j = \frac{2(z_{\bullet j}^* - z_{\bullet j}) - \lambda_{\bullet}}{n} \text{ and } n \mu_{\bullet} = 2(z_{\bullet\bullet}^* - z_{\bullet\bullet}) - m \lambda_{\bullet}.$$

So,  $\lambda_{\bullet}$  and  $\mu_{\bullet}$  are arbitrary but linked by  $m \lambda_{\bullet} + n \mu_{\bullet} = 2(z_{\bullet\bullet}^* - z_{\bullet\bullet})$ . One can take  $\lambda_{\bullet} = \mu_{\bullet}$ ,

what gives:  $\lambda_{\bullet} = \mu_{\bullet} = 2 \frac{z_{\bullet\bullet}^* - z_{\bullet\bullet}}{m+n}$ ; hence:

$$\lambda_i = 2 \frac{(m+n)(z_{i\bullet}^* - z_{i\bullet}) - (z_{\bullet\bullet}^* - z_{\bullet\bullet})}{m(m+n)} \text{ and } \mu_j = 2 \frac{(m+n)(z_{\bullet j}^* - z_{\bullet j}) - (z_{\bullet\bullet}^* - z_{\bullet\bullet})}{m(m+n)}$$

which reported in (13) gives:  $\hat{z}_{ij}^A = \frac{z_{i\bullet}^* - z_{i\bullet}}{m} + z_{ij} + \frac{z_{\bullet j}^* - z_{\bullet j}}{n} - \frac{z_{\bullet\bullet}^* - z_{\bullet\bullet}}{nm}$ .

### 1.1.1 Solution of minimization of information lost

For vectors, the Lagrangian writes as:

$$L = \sum_{i=1}^n \hat{x}_i^M \log \frac{\hat{x}_i^M}{x_i} + \lambda \left( x_{\bullet}^* - \sum_{i=1}^n \hat{x}_i^M \right)$$

The first derivatives are all zero at optimum:

$$\frac{dL}{d\hat{x}_i^M} = 0 \text{ for all } i \Leftrightarrow \log \frac{\hat{x}_i^M}{x_i} + 1 - \lambda = 0 \text{ for all } i \Leftrightarrow$$

$$(13) \hat{x}_i^M = e^{\lambda-1} x_i \text{ for all } i$$

$$\frac{dL}{d\lambda} = 0 \Leftrightarrow x_{\bullet}^* = \sum_{i=1}^n \hat{x}_i^M$$

So, by replacing  $\hat{x}_i^M$  from (11) in this expression:  $x_{\bullet}^* = e^{\lambda-1} x_{\bullet} \Leftrightarrow \lambda = \log \frac{x_{\bullet}^*}{x_{\bullet}} + 1$ ; then by

carrying this value into (11):  $\hat{x}_i^M = a x_i$  for all  $i$ .

For matrices, the Lagrangian writes as:

$$(14) \quad L = \sum_{i=1}^n \sum_{j=1}^m \hat{z}_{ij}^M \log \frac{\hat{z}_{ij}^M}{z_{ij}} + \lambda_i \left( z_{i\bullet}^* - \sum_{j=1}^m \hat{z}_{ij}^M \right) + \mu_j \left( z_{\bullet j}^* - \sum_{i=1}^n \hat{z}_{ij}^M \right)$$

$$\frac{dL}{d\hat{z}_{ij}^M} = 0 \Leftrightarrow 1 + \log \frac{\hat{z}_{ij}^M}{z_{ij}} - \lambda_i - \mu_j = 0 \Leftrightarrow$$

$$(15) \quad \hat{z}_{ij}^M = z_{ij} \exp\left(-\left(\lambda_i + \mu_j\right)\right) \text{ for all } i, j$$

Then,

$$\frac{dL}{d\lambda_i} = 0 \Leftrightarrow z_{i\bullet}^* = \sum_{j=1}^m \hat{z}_{ij}^M \text{ and by inserting (15):}$$

$$z_{i\bullet}^* = \exp(-\lambda_i) \sum_{j=1}^m z_{ij} \exp\left(-\left(1 - \mu_j\right)\right) \Leftrightarrow \exp(-\lambda_i) = \frac{z_{i\bullet}^*}{\sum_{j=1}^m z_{ij} \exp\left(-\left(1 - \mu_j\right)\right)} \text{ for all } i$$

And,

$$\frac{dL}{d\mu_j} = 0 \Leftrightarrow z_{\bullet j}^* = \sum_{i=1}^n \hat{z}_{ij}^M \text{ and by inserting (15):}$$

$$z_{\bullet j}^* = \exp\left(-\left(1 - \mu_j\right)\right) \sum_{i=1}^n z_{ij} \exp(-\lambda_i) \Leftrightarrow \exp\left(-\left(1 - \mu_j\right)\right) = \frac{z_{\bullet j}^*}{\sum_{i=1}^n z_{ij} \exp(-\lambda_i)} \text{ for all } j$$

Then by reporting both it into (15):

$$\hat{z}_{ij}^M = \frac{z_{i\bullet}^*}{\sum_{j=1}^m z_{ij} \exp\left(-\left(1 - \mu_j\right)\right)} z_{ij} \frac{z_{\bullet j}^*}{\sum_{i=1}^n z_{ij} \exp(-\lambda_i)}$$

By denoting  $p_i = \exp(-\lambda_i)$  and  $q_j = \exp\left(-\left(1 - \mu_j\right)\right)$ , one obtains  $\hat{z}_{ij}^M = p_i z_{ij} q_j$  for all  $i, j$  with

$$p_i = \frac{z_{i\bullet}^*}{\sum_{j=1}^m z_{ij} q_j} \text{ for all } i \text{ and } q_j = \frac{z_{\bullet j}^*}{\sum_{i=1}^n z_{ij} p_i} \text{ for all } j.$$

Actually, if  $z_{ij} = 0$ , the function (14) is not defined:  $\lim_{x \rightarrow 0} \log \frac{1}{x} = +\infty$ .<sup>20</sup> Unfortunately, this case often occurs: a matrix  $\mathbf{Z}$  in the real world (as an input-output table) does contain many zeros

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<sup>20</sup> Many thanks to Christian Michelot for this remark.



and it is not realistic to impose  $\mathbf{Z}$  to be strictly positive. It is computationally annoying. This is why Geoffrey Hewings has attempted to solve the problem in its Ph.D. dissertation (1969) by replacing the zeros by very small values, arguing that these zeros are often caused by the rounding of small values. It is a great idea for real interindustrial and interregional tables because these tables are practically indecomposable into two (or more) independent blocks: replacing a zero by a small value does not change much the picture even if it is a true zero that is replaced by a small value. For other types of tables, it could not work, for example for international tables where two blocks of countries are in autarky each other, or for social tables where two groups of individual do not exchange messages. In these exceptional cases, the matrix can be decomposed into two blocks  $\mathbf{A}_{11}$  and  $\mathbf{A}_{22}$  (there are no exchanges between the two blocks), and one has two dominant eigenvalues that determine two different growth rates for each block:

$$\mathbf{Z} = \begin{array}{|c|c|} \hline \mathbf{A}_{11} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{A}_{22} \\ \hline \end{array}$$

So if a zero not placed in a block is replaced by a small term as in:

$$\mathbf{Z} = \begin{array}{|c|c|} \hline \mathbf{A}_{11} & \begin{array}{cc} \mathbf{0} & \mathbf{0} \\ \epsilon & \\ \mathbf{0} & \mathbf{0} \end{array} \\ \hline \mathbf{0} & \mathbf{A}_{22} \\ \hline \end{array}$$

the system will have only one dominant eigenvalue and only one growth rate, instead of two: this completely changes the picture even if the equalization of the growth rates into a new unique one could take much time (i.e., many iterations). Hewings procedure could allow finding a solution to the adjustment problem even when this solution would not exist normally: while a zero is projected as a zero, a small value can be projected a large value.

However, on a theoretical point of view, there is no difficulty if  $z_{ij} = 0$  for some  $i$  and  $j$ . Equation (14) can be defined for  $z_{ij} \neq 0$ , for all  $i, j$ , what allows to derive equation (15). Then, one is able to calculate the limit value of  $\hat{z}_{ij}^M$  when  $z_{ij} \rightarrow 0$  for any  $i, j$ :  $\hat{z}_{ij}^M \rightarrow 0$  when  $z_{ij} \rightarrow 0$  for any  $i, j$ . Remark that this difficulty is specific to the minimization of information lost and does not occur when the model is derived from entropy maximization.

## 1.1 Annex 2: examples

It could be strange to think about the foundations of the linear model, as this one is so much familiar to everyone and seems to be known by every scholar. The applications are obvious and very common, for example:

| Given a consumption function, what is the consumption of each commodity if the consumer's revenue increases, the utility consumer's function being unknown?

| Given a production function with non substitutable factors (fixed coefficients following Leontief), what is the vector of inputs that correspond to a larger production, the production function being unknown?

### 1.1.1 Consumption function

The present vector of consumption of an agent is (units are in value):  $\mathbf{x}' = (10 \ 65 \ 100 \ 25 \ 200)$ . The wealth is  $w = 400$ .

Now, the agent's wealth becomes  $w = 600$ . Following the linear model, i.e., the information-lost-minimization principle, the new consumption vector turns out to be

$\hat{\mathbf{x}}^M = (15 \ 97.5 \ 150 \ 37.5 \ 300)$  with  $a = \frac{600}{400} = 1.5$ . Following the distance-minimization

principle the consumption vector becomes  $\hat{\mathbf{x}}^A = (50 \ 105 \ 140 \ 65 \ 240)$  with  $b = \frac{600 - 400}{5} = 40$ .

The Euclidean distance between  $\hat{\mathbf{x}}^A$  and  $\mathbf{x}$  is 89.44 while the Euclidean distance between  $\hat{\mathbf{x}}^M$  and  $\mathbf{x}$  is higher, at 117.21. When  $w$  tends toward infinite,  $\hat{\mathbf{x}}^A$  tends to have all its components equal to  $\frac{w}{5}$ , what is economically nonsense, while the proportions inside  $\hat{\mathbf{x}}^M$  remain fixed.

If the agent's wealth goes to  $w = 200$ , some negative terms will appear with distance minimization:  $b = \frac{200-400}{5} = -40$ ,  $\hat{\mathbf{x}}^A = (-30 \ 25 \ 60 \ -15 \ 160)$  and the agent has a negative consumption of commodities 1 and 4. With the information-lost-minimization principle,  $a = \frac{200}{400} = 0.5$  and  $\hat{\mathbf{x}}^M = (5 \ 32.5 \ 50 \ 12.5 \ 100)$  is not negative. Nevertheless  $\hat{\mathbf{x}}^A$  that is the closest to  $\mathbf{x}$ . The distances are unchanged (they depend only on  $|b|$  and  $a$ , here unchanged by comparison to the case  $w = 600$ ): between  $\hat{\mathbf{x}}^A$  and  $\mathbf{x}$  it is again 89.44 and between  $\hat{\mathbf{x}}^M$  and  $\mathbf{x}$  it is 117.21.

### 1.1.1 Production function

The vector of inputs of a sector is  $\mathbf{x}' = (80 \ 20 \ 40 \ 60 \ 30 \ 70)$  with an output of 400 (the value added is equal to 100).

Now, the new output is 700 with a value added of 175 (assuming the ratio added value / output to be stable and equal to 0.25). With multiplicative methods, the technical coefficients remain stable in accordance with Leontief:  $\boldsymbol{\theta}^M = (.2 \ .05 \ .1 \ .15 \ .075 \ .175)$  and the new vector of intermediary consumption becomes  $\hat{\mathbf{x}}^M = (140 \ 35 \ 70 \ 105 \ 52.5 \ 122.5)$  with

$a = \frac{700-175}{400-100} = \frac{525}{300} = 1.75 = \frac{700}{400}$ . The Euclidean distance between  $\hat{\mathbf{x}}^M$  and  $\mathbf{x}$  is 100.06.

However, following the criteria of distance minimization the new output vector becomes  $\hat{\mathbf{x}}^A = (117.5 \ 57.5 \ 77.5 \ 97.5 \ 67.5 \ 107.5)$  with  $b = \frac{525-300}{6} = 37.5$ . The Euclidean distance between  $\hat{\mathbf{x}}^A$  and  $\mathbf{x}$  is lower at 91.86. The new technical coefficients are  $\boldsymbol{\theta}^A = (.168 \ .082 \ .111 \ .139 \ .096 \ .154)$ .

As it can be seen, the technical coefficients cannot be stable with the distance principle: at infinite, they tend to be all equal to  $\frac{.75}{6} = .125$ , that is,  $\boldsymbol{\theta}^A \rightarrow (.125 \ .125 \ .125 \ .125 \ .125 \ .125)$ , those above 0.125 being decreasing up to this value, those under .125 increasing up to it, what is economically non sense. However, the technical coefficients  $\boldsymbol{\theta}^M$  remain fixed with the linear model.

If the output goes down to 200, the value-added becomes equal to 50;  $a = .5$  and  $\hat{\mathbf{x}}^M = (40 \ 10 \ 20 \ 30 \ 15 \ 35)$  but  $b = \frac{150-300}{6} = -25$  and  $\hat{\mathbf{x}}^A = (55 \ -5 \ -15 \ 35 \ 5 \ 45)$ : the intermediate consumption of input 2 becomes negative even if vector  $\hat{\mathbf{x}}^A$  is closer to  $\mathbf{x}$  than vector  $\hat{\mathbf{x}}^M$  :  $d(\hat{\mathbf{x}}^M, \mathbf{x}) = 66.71$  and  $d(\hat{\mathbf{x}}^A, \mathbf{x}) = 61.24$ .

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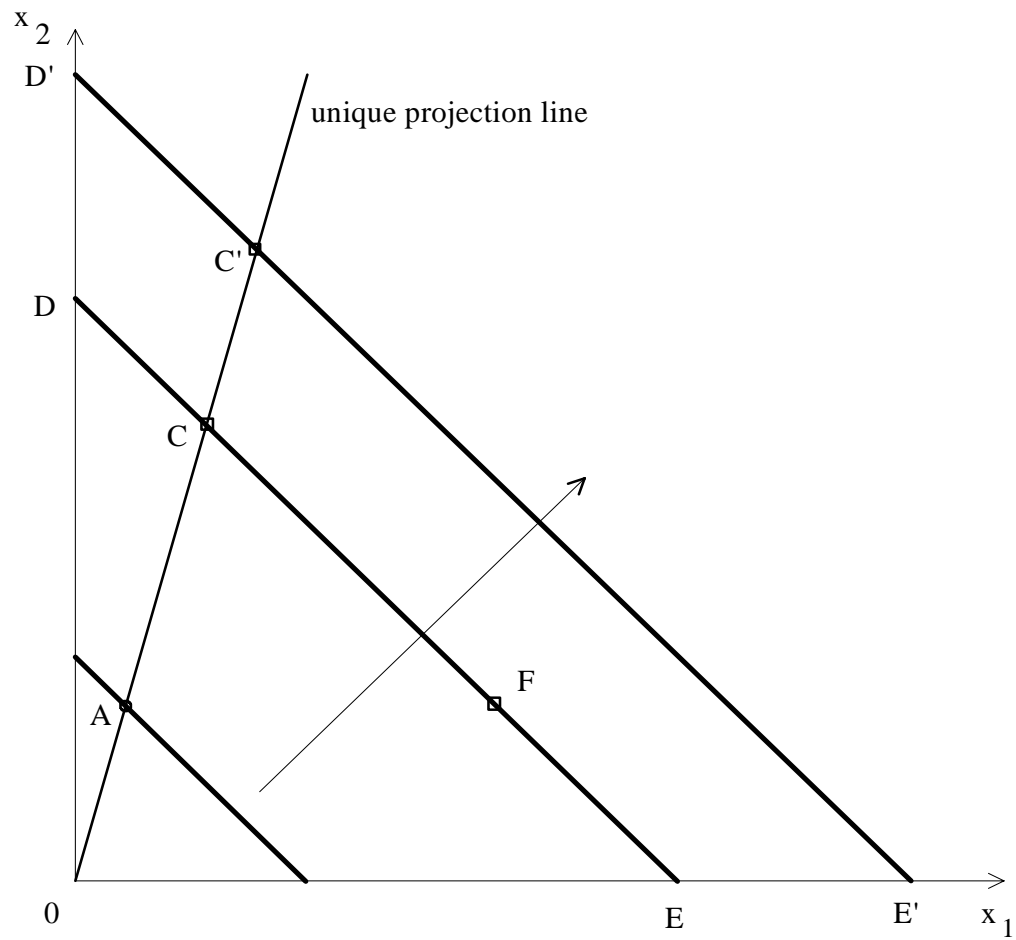


Figure 2. Linear projection in two dimensions  
(circle: A, initial point; squares: C and C', projected points)



